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THE STABILITY OF QUASI-TRANSVERSE SHOCK WAVES IN ANISOTROPIC ELASTIC MEDIA†

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The stability of weak quasi-transverse shock waves in a weakly anisotropic elastic medium with respect to arbitrarily oriented perturbations is investigated in the linear approximation. It is shown that fast quasi-transverse shock waves are stable. © 2001 Elsevier Science Ltd. All rights reserved.

The problem of the stability of quasi-transverse shock waves is of special interest in view of the non-uniqueness of the solutions of homogeneous self-similar problems for weakly anisotropic elastic media [1, 2]. It has been shown that the non-uniqueness is due to the existence of "metastable" shock waves, for which the conservation laws do not prevent breakdown into a system of waves propagating with different velocities. The stability of quasi-transverse shock waves with respect to non-linear perturbations of the same direction, which remains a shock wave after introducing a viscosity into the equation (the Voigt model), was investigated using numerical experiments in [3, 4] and it was shown that "metastable" shock waves are stable with respect to perturbations whose amplitude is with the shock wave amplitude.

In this paper, we consider the stability of shock waves in anisotropic elastic media with respect to arbitrarily oriented perturbations using the smallness of their amplitude, which enables us to write and investigate simplified equations for the perturbations.

1. SIMPLIFIED EQUATIONS TO DESCRIBE WAVES HAVING SIMILAR ORIENTATIONS

We will first consider one-dimensional quasi-transverse small-amplitude waves propagating in a uniform medium with normals directed in the positive direction of the x axis of a Cartesian system of coordinates, i.e. solutions which depend on x and t. These solutions can be described using the equations [5]

$$\frac{\partial u_\alpha}{\partial t}+\frac{\partial}{\partial x}\frac{\partial R}{\partial u_\alpha}=0, \quad \alpha=1,2 \tag{1.1}$$

$$R(u_1,u_2)=\frac{1}{2}(f-g)u_1^2+\frac{1}{2}(f+g)u_2^2-\frac{1}{4}\varkappa(u_1^2+u_2^2)^2, \quad u_\alpha=\frac{\partial w_\alpha}{\partial x}$$

Here x is the Lagrangian coordinate, w_α are the components of the displacement vector in the direction of the x_1 and x-axes, orthogonal to the x axis and orthogonal to one another, and f, g and $\varkappa$ are coefficients characterizing the elastic medium (g and $\varkappa$ are coefficients characterizing the anisotropy and non-linearity of the medium, respectively). The quantity f represents the velocity of propagation of the perturbations when there is no non-linearity and no anisotropy. This quantity changes under a Galilean transformation.

For the solutions of Eqs (1.1) the relations on discontinuities follow from the conditions of conservation of the flux of the transverse components of the momentum

$$[\partial R(u_1,u_2)/\partial u_\alpha]=W[u_\alpha] \tag{1.2}$$

The square brackets denote jumps in the quantities they enclose: $[u_\alpha]=u_\alpha^+-u_\alpha^-$, where u_α^- is the value in front of the shock-wave front while u_α^+ is the value behind the front. Evolutionary shock waves for which the following inequalities are satisfied

$$W\geqslant C_2^-, \quad C_1^+\leqslant W\leqslant C_2^+ \tag{1.3}$$

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will be called fast shock waves. Here C_1 and C_2 are the slow and fast velocities of the characteristics of system (1.1) The plus superscript will be omitted henceforth in many (rather obvious) cases.

We will obtain a simplified system of equations for the two-dimensional motions of an elastic medium, for which all the variable quantities depend on the time t and the x coordinate and also depend only slightly on the Lagrangian coordinate $y = x_2$.

Bearing in mind that, small-amplitude, waves interact effectively only in the case of close wave numbers, we will consider the behaviour of waves with normals close to the positive direction of the x axis. The equations which describe these waves will be obtained by the method used previously to obtain the Kadomtsem–Petviashvili [6] and Khokhlov–Zabolotskii [7] equations. The main assumption is that, in view of both the small non-linearity and anisotropy, and also the small deviation of a wave from the one-dimensional one, which depend only on x and t, the terms describing the influence of the above-mentioned effects should enter into the final equations independently. Hence, we can supplement Eqs (1.1) with terms which take into account the small non-uniformity, and are obtained assuming that there is no non-linearity and anisotropy.

In the linear isotropic case, both components of the transverse perturbations must satisfy the same wave equations [8]. Assuming that the transverse perturbations depend on x, y and t as $\exp(i(kx + ly - \Omega t))$, we obtain

$$\Omega^2 = f^2(k^2 + l^2) \tag{1.4}$$

where a is the velocity of propagation of transverse perturbations.

Assuming that $l^2 \ll k^2$, retaining the principal term in l^2/k^2 and changing to a system of coordinates which moves along the x axis with a velocity W, such that $b = f - W \ll f$, we obtain

$$k\omega = bk^2 + fl^2/2, \quad \omega = \Omega - kU \tag{1.5}$$

As in [6, 7], comparing relations (1.5) and (1.1) we conclude that the equations describing weakly non-uniform quasi-transverse waves in a weakly anisotropic elastic medium have the form

$$\frac{\partial}{\partial x}\left[\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x}\frac{\partial R(u_1, u_2)}{\partial u_\alpha}\right] = -\frac{f}{2}\frac{\partial^2 u_\alpha}{\partial y^2}, \quad \alpha = 1, 2 \tag{1.6}$$

If the medium is viscoelastic, the effect of viscosity, like the non-linearity and other factors mentioned above, being small in the solutions investigated, the viscous terms can be written in the one-dimensional isotropic approximation, so that the equations for the quasi-transverse waves take the form

$$\frac{\partial}{\partial x}\left[\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x}\frac{\partial R(u_1, u_2)}{\partial u_\alpha} - \nu\frac{\partial^2 u_\alpha}{\partial x^2}\right] = -\frac{f}{2}\frac{\partial^2 u_\alpha}{\partial y^2} \tag{1.7}$$

where ν is the kinematic coefficient of viscosity.

In Eqs (1.6) and (1.7) $R(u_1 u_2)$ is the expression given in (1.1). However, in what follows we are going to use these equations in a system of coordinates moving with the shock wave, with a velocity W close to f. The corresponding Galilean transformation leads to the replacement of f by $b = f - W$ in the expression for $R(u_1, u_2)$.

2. SMALL PERTURBATIONS

To investigate the stability of a shock wave we need to have available solutions of the linearized equations behind and in front of the shock wave. We will linearize Eqs (1.6) about the uniform state $u_\alpha = u_\alpha^0$ and write them in the form

$$\frac{\partial u_\alpha}{\partial t} + R_{\alpha\beta}^0\frac{\partial u_\beta}{\partial x} = -\frac{1}{2}f\frac{\partial v_\alpha}{\partial y}, \quad \frac{\partial v_\alpha}{\partial x} = \frac{\partial u_\alpha}{\partial y} \tag{2.1}$$

$$R_{\alpha\beta}^0 = \left(\frac{\partial^2 R(u_1, u_2)}{\partial u_\alpha \partial u_\beta}\right)_{u_\gamma = u_\gamma^0}, \quad v_\alpha = \frac{\partial w_\alpha}{\partial y}, \quad \alpha, \beta, \gamma = 1, 2$$

Equations (2.1) were obtained from Eqs (1.6) integrated with respect to x, taking into account the assumption that the derivatives of the quantities u_α vanish at infinity. Substituting into (2.1) expressions of the form

$$u_\alpha = \hat{u}_\alpha \exp(i(kx + ly - \omega t)) \quad v_\alpha = \hat{v}_\alpha \exp(i(kx + ly - \omega t)) \tag{2.2}$$

we obtain from the last pair of equations (2.1)

$$\hat{v}_\alpha = \hat{u}_\alpha l / k \tag{2.3}$$

and from the first pair of the equations, taking relation (2.3) into account, we obtain

$$(R_{11} - c)\hat{u}_1 + R_{12}\hat{u}_2 = 0, \quad R_{12}\hat{u}_1 + (R_{22} - c)\hat{u}_2 = 0 \tag{2.4}$$

Here, we have introduced the notation

$$c = \frac{\omega}{k} - \frac{f}{2}\frac{l^2}{k^2} \tag{2.5}$$

as a result of which Eqs (2.4) are identical with the equations describing the propagation of one-dimensional perturbations. Here $c = c_{1,2} = C_{1,2} - W$ are the velocities of the characteristics of the one-dimensional system, which are found from the quadratic equation obtained by equating the determinant of system (2.4) to zero. These quantities are real since the matrix $R_{\alpha\beta}$ is symmetrical. The numbering is chosen so that $c_1 < c_2$.

The quantities $\hat{u}_\alpha$ are found from (2.4) for each value of $c = c_{1,2}$. We will denote these quantities by $u_\alpha^{(1)}$ and $u_\alpha^{(2)}$. If we assume l and ω to be known, the two values of k can be obtained from Eq. (2.5), which is quadratic in k. Here, we will assume that the quantities $k_1 \leqslant k_2$ correspond to c_1, while the quantities $k_3 \leqslant k_4$ correspond to c_2. The quantities $\hat{v}_\alpha$ are found from Eqs (2.3). Hence, for specified l and ω we have four solutions of the form (2.2). System of equations (2.3) and (2.4) define 4 eigenvectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, corresponding to k_1, k_2, k_3 and k_4. The components of these eigenvectors u_1, u_2, v_1, v_2 are such that

$$\begin{aligned} \mathbf{r}_i &= (u_1^{(1)}, u_2^{(1)}, u_1^{(1)} l / k_i, \quad u_2^{(1)} l / k_i), \quad i = 1, 2 \\ \mathbf{r}_j &= (u_1^{(2)}, u_2^{(2)}, u_1^{(2)} l / k_j, \quad u_2^{(2)} l / k_j), \quad j = 3, 4 \end{aligned} \tag{2.6}$$

We will investigate which of these solutions correspond to waves arriving at and departing from the shock wave.

Consider the velocity diagram, connecting it with the velocity of the unperturbed shock wave (see Fig. 1). Along the horizontal axis we have plotted the x-components of the velocities (normal to the

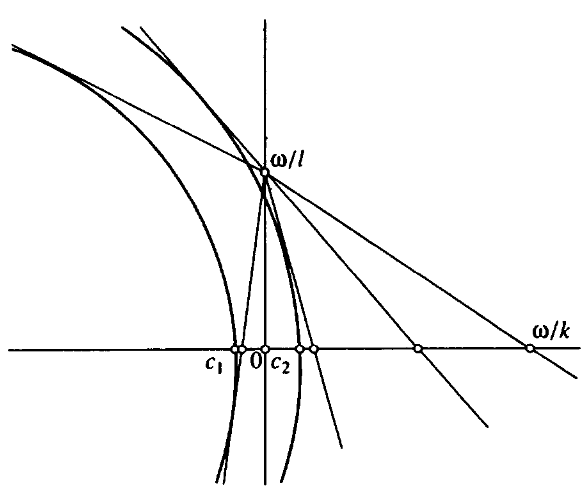

Fig. 1

shock wave), and along the vertical axis we have plotted the y-components (tangential to the shock wave). Along the vertical axis we have plotted the value of ω/l (l and ω are assumed to be real here). Moreover, in the figure we show a diagram of the group velocities of small perturbations (representing, at $t = 1$, the position of small perturbation fronts, which develop from a point perturbation situated at the point O at $t = 0$). These curves are similar to circles with radii close to a and centres lying at the point $(-U, 0)$.

To find the values of ω/k it is sufficient to draw tangents to the group-velocity diagrams from the point $(0, \omega/l)$. These tangents intersect the horizontal axis at the point $(\omega/k, 0)$. If we consider waves in the region $x < 0$ (behind the shock wave), the perturbations arriving at the shock wave will be those which correspond to points lying to the right of the vertical axis in the figure (and for a specified value of ω/l the points of contact mentioned above). If we consider the region $x > 0$ (in front of the shock wave), we obtain the converse.

In the figure we show the situation behind a fast shock wave, when the value of ω/l is fairly high. In this case there are four points of contact, to which four real values of k correspond. It is obvious that in this situation there is a single arriving wave which corresponds to c_2 and a lower value of ω/k, that is $k = k_4$ (the larger of the values k_3 and k_4). Hence, in the case considered the departing waves correspond to $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. Since the fast shock wave propagates over the state ahead of it with a velocity greater than both values of c, there is no departing perturbations in front of the shock wave.

When ω/l is reduced (this quantity, as before, is assumed to be real), the two points of contact corresponding to c_2 may disappear. In this case two real values of k disappear (after these points merge). Since we are considering analytical relations this means that there are two complex-conjugate values of k. One of these may be assumed to correspond to an arriving perturbation and the other to a departing one. The departing perturbation tends to zero at infinity, i.e. when $x < 0$ we have $\operatorname{Im} k < 0$ for departing perturbations. We will denote this quantity by k_3. Then, as in the case of real k, the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ will correspond to perturbations departing from the shock wave. Note that the separation into arriving and departing perturbations has been carried out for real values of ω, which is permissible because of the stability of the homogeneous solutions of Eqs (1.6).

3. BOUNDARY CONDITIONS FOR PERTURBATIONS BEHIND A FAST SHOCK WAVE

We will assume that the unperturbed position of the shock wave is $x = 0$. Suppose the perturbed position is given by the equation

$$x = \xi(y,t) = \hat{\xi}\exp(i(ly - \omega t))$$

where $\hat{\xi}$ is a small quantity. In the linear approximation the vector of the normal $\mathbf{n}$ and the tangential vector $\boldsymbol{\tau}$ have the components $\{1, -il\hat{\xi}\}$ and $\{il\hat{\xi}, 1\}$, respectively. On the shock wave the displacements w_α remain continuous. This can be written in the form of an equation in which the derivative of the discontinuity of displacement along the tangent is equated to zero

$$(il\hat{\xi}[u_\alpha] + [v_\alpha])_{x=\xi} = 0 \tag{3.1}$$

We will introduce notation for the discontinuity of the derivative with respect w_α along the normal

$$([u_\alpha] - il\hat{\xi}[v_\alpha])_{x=\xi} = a_\alpha \tag{3.2}$$

When writing the left-hand sides of Eqs (3.1) and (3.2) we have used expressions for the components of the vectors $\mathbf{n}$ and $\boldsymbol{\tau}$ and also the fact that u_α and v_α are derivatives of w_α with respect to the spatial coordinates.

As can be seen from (3.1), the second term on the left-hand side of (3.2) is proportional to $\hat{\xi}^2$, and hence can be neglected in the linear approximation. Since the quantity $\xi[\partial u_\alpha/\partial x]$ is also a second-order infinitesimal, for $x = 0$ condition (3.2) can be written in the following form

$$[u_\alpha]_{x=0} = a_\alpha \tag{3.3}$$

From the same considerations Eq. (3.1) can be assumed satisfied when $x = 0$.

The values of a_α may differ from their unperturbed values a_α^0 due to the change in the direction of the shock wave, defined by the quantity $\partial\xi/\partial y = il\hat{\xi}$ and due to a change in its velocity $\delta W = \partial\xi\partial t = -i\omega\hat{\xi}$.

To find how a_α depends on $\partial\xi/\partial y$ we will consider the solutions of Eqs (1.7), which take the viscosity into account, in the form of an inclined travelling wave resembling a smoothed step of the form $u_\alpha = u_\alpha(\zeta)$, $\zeta = x - \lambda y - Wt$. The quantities a_α are represented in this case as the difference between the values of u_α at $\zeta = \pm\infty$. We obtain from (1.7)

$$-Wu_\alpha'' + (\partial R/\partial u_\alpha)'' - \mu u_\alpha''' = -f\lambda^2 u_a''/2 \tag{3.4}$$

It can be seen that if we introduce a new shock-wave velocity given by the formula $W_1 = W - f\lambda^2/2$, this equation takes the form of an equation for a shock wave normal to the x axis moving with velocity W_1. Since the quantities $\lambda = \partial\xi/\partial y$ under consideration are infinitesimal, the difference between W and W_1 can be neglected and we can assume that a small inclination of the shock wave does not lead to changes in the values of $[u_\alpha]$ in the linear approximation. Hence, in the approximation adopted, the right-hand side of (3.3) may change only due to a change in the velocity W of the shock wave. If we assume that there are no perturbations in front of the shock wave (when $x > 0$), we obtain the following relations for the perturbations of the quantities u_α and v_α when $x = -0$

$$\tilde{u}_\alpha = -a_\alpha' i\omega\hat{\xi}, \quad \tilde{v}_\alpha = -a_\alpha^0 il\hat{\xi}$$

$$a_\alpha' = \partial a_\alpha/\partial W = \partial u_\alpha/\partial W, \quad a_\alpha^0 = [u_\alpha] \tag{3.5}$$

The quantities a_α' represent the components of the vector tangential to the shock adiabatic curve and a_α^0 are the components of the vector from the initial point of the shock adiabatic curve to the point characterizing the state behind the shock wave.

4. THE CONDITIONS FOR EIGENFUNCTION TO EXIST

It was shown in [9] that instability of the discontinuities may manifest themselves only in the form of perturbations which grow exponentially with time. Since we are investigating the stability of fast shock waves here, departing perturbations only occur behind the shock wave ($x > 0$). If we consider perturbations of the form $\exp(i(ly - \omega t))$, these perturbations must be represented, for $x = 0$, by a linear combination of eigenvectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, represented by formulae (2.6). This linear combination must satisfy boundary conditions (3.5). The last condition can also be formulated as the possibility of expanding the vector represented by (3.5) into vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ or, otherwise, as the condition that the vector (3.5) should be orthogonal to the vector orthogonal to the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ which we will denote by $\mathbf{r}$. Using the orthogonality of the two-dimensional vectors $u_1^{(1)}$, $u_2^{(1)}$ and $u_1^{(2)}$ and $u_2^{(2)}$ (which follows from the symmetry of the matrix $\|R_{\alpha\beta}\|$), we obtain

$$\mathbf{r}_* = (u_1^{(2)}, u_2^{(2)}, -u_1^{(2)}k_3/l, -u_2^{(2)}k_3/l) \tag{4.1}$$

The above condition for the vectors (3.5) and (4.1) to be orthogonal can be written in the vector form

$$(\mathbf{a}^0 - \mathbf{a}'\omega/k_3), \ \mathbf{u}^{(2)} = 0 \tag{4.2}$$

$$\mathbf{a}^0 = (a_1^0, a_2^0), \quad \mathbf{a}' = (a_1', a_2'), \quad \mathbf{u}^{(2)} = (u_1^{(2)}, u_2^{(2)})$$

(the scalar product is denoted by a dot). From this equation we determine ω/k_3, which enables us below to find ω, using relation (2.5).

The relation between a_α' and a_α^0 can be obtained by differentiating the relations on the discontinuity (1.2) with respect to W, assuming u_α^- is constant, and taking into account the fact that $W = 0$ for the unperturbed wave. We have

$$R_{\alpha\beta}^0 a_\beta' = a_\alpha^0 \tag{4.3}$$

To simplify Eq. (4.2) we will introduce at the point u_1, u_2, which represents the state of the unperturbed shock wave, a new system of coordinates q_1, q_2 with axes directed along the eigenvectors of the matrix $\|R_{\alpha\beta}\|$, corresponding to c_1 and c_2. In this system of coordinates the matrix $\|R_{\alpha\beta}\|$ will be a diagonal matrix with quantities c_1 and c_2 on the principal diagonal, the vector $\mathbf{u}^{(2)}$ will have components (0, 1), while for the components A_2' and A_2^0 of the vectors a' and a^{0-} along the q_2 axis we obtain from relation (4.3)

$$c_2 A_2' = A_2^0 \tag{4.4}$$

Equation (4.2) yields

$$A_2^0 - A_2'\omega / k_3 = 0$$

Hence also from relation (4.4) we have

$$\omega / k_3 = c_2$$

It then follows from Eq. (2.5) that

$$l^2 / k_3^2 = 0$$

This means that when $l \neq 0$ it is impossible to set up an eigenfunction representing a linear combination of departing waves. The case $l = 0$ (the one-dimensional case) must be considered separately since it was previously assumed that $l \neq 0$. One-dimensional interactions were investigated fairly fully in [3], whence we can conclude that the shock wave is stable with respect to linear perturbations of the same orientation everywhere except for the Jouguet point where $W = c_1$.

Hence, fast shock waves with $W \neq c_1$ are stable to arbitrary linear perturbations.

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